In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

7/25/68

THE LIFTING AND GENERALIZED EXTENSION PROBLEMS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Eduardo Ramon Bastida

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

August, 1969

THE LIFTING AND GENERALIZED EXTENSION PROBLEMS

Approved:

Chairman

Date approved by Chairman: Aug. 21, 1969

ACKNOWLEDGMENTS

I wish to express my gratitude to Dr. Roger D. Johnson for his guidance, patience, and interest during the past two years. I am also grateful to Dr. George Cain and Dr. Ian Gatland for reading the manuscript and for their helpful suggestions. Finally, I am grateful to Mrs. Betty Sims for her expert typing.

TABLE OF CONTENTS

SUMMARY

The purpose of this study is to examine two important problems of topology, the extension problem and the lifting problem. We may state the extension problem as follows: given a subset A of a space X and a continuous function f: $A \to Y$, does there exist a continuous F: $X \to Y$ such that $F/A = f$? The lifting problem reads as follows: given two continuous functions f: $X \to Y$ and p: $\tilde{X} \to Y$, does there exist a continuous function f': $X \to \tilde{X}$ such that $pf' = f$?

We shall also be concerned with generalizations of the extension problem, such as: given two continuous functions f: $X \to Y$ and g: $X \to Z$, does there exist a continuous h: $Y \to Z$ such that $hf = g$? In the following chapters this particular generalization will be referred to as the generalized extension problem. Further generalizations of both, the extension and lifting problems will be discussed in Chapter II.

Chapter 0 is a brief summary of some concepts and results of topology which are of particular importance to the subsequent text. References are given to show where detailed discussions of these results may be found.

Chapter I is divided into two sections. Section 1 is devoted to the generalized extension problem mentioned above. We show that, under certain conditions, if the map f is compact, then f has an inverse (i.e., a map s such that $s \circ f$ and $f \circ s$ are the identity maps on X and Y, respectively). We then define $h = gs$ from which it follows that

hf = g. We also show that, if f is an identification map (see [1]), and if the map g is suitably restricted, then there exists a map h such that hf = g.

In Section 2 we concentrate on the lifting problem. It is shown that, if the map p is a covering projection and the spaces considered are path connected, the lifting problem admits a solution. This leads us, in a natural manner, into the concept of a lifting function. A lifting function for p is a function which sends continuous paths in Y to continuous paths in $\tilde{X}$. We show that, under fairly general conditions, if the map p has a lifting function, then the map f has a lifting f' to $\tilde{X}$ provided that a certain relation exists between the closed continuous paths in X and the closed continuous path in $\tilde{X}$. This last result enables us to prove a lifting theorem and some related results treated by E. Spanier in [3]. It is felt that the approach taken in this thesis is somewhat more intuitive and direct than the one in [3].

In Chapter II, algebraic concepts are introduced, on the basis of which further generalizations of the extension and lifting problems are considered. The first section of this chapter introduces the first homotopy group of a topological space, also known as the fundamental group of the space. This fundamental group will be the main tool used when transforming the topological problem into an algebraic one. In Sections 2 and 3 the homotopy extension property and the homotopy lifting property are defined. These change lifting and extension problems of a map into a problem of the homotopy class of the map. Also in

Section 3 the "lifting theorem" (see [3], 2.4.5) is proved, using the results obtained in Section 2 of Chapter I. This lifting theorem gives a necessary and sufficient condition for a solution of the lifting problem in terms of the fundamental groups of the spaces involved when the map p: $\tilde{X} \to Y$ is a fibration.

The lifting theorem in Section 3 of Chapter II establishes the relevance of fibrations to the lifting problem. It is then desirable to consider such maps in more detail. In Chapter III, by the use of the methods in [1], it is shown that, if the space Y is paracompact, the property that a map be a fibration is a local matter. Further discussions of local fibrations may be found in [1] and [3].

CHAPTER 0

INTRODUCTION

Before dealing with the subject matter of this study, a brief summary of some concepts and results of general topology is in order.

Some of the results are not standard, but are of particular importance in the subsequent development. In each case a reference is given.

Let X and Y be topological spaces and let X^Y be the space of all continuous maps from Y into X with the compact open topology defined as follows.

Definition 1. If X and Y are spaces, $A \subset Y$ and $B \subset X$, then (A,B) denotes the subset of X^Y consisting of all maps f such that $f(A) \subset B$.

Definition 2. If X and Y are spaces, the <u>compact open topology</u> in X^Y is that one having as a subbasis all sets (K,U), where $K \subset Y$ is compact and $U \subset X$ is open.

Definition 3. For any two spaces X and Y, the map $E\colon X^Y \times Y \to X$ defined by $E(f,y) = f(y)$ is called the <u>evaluation map</u>.

The two following theorems may be found in [2] (Chapter XII, Sections 2 and 3).

Theorem 1. If Y is a locally compact Hausdorff space, the evaluation map $E\colon X^Y \times Y \to X$ is continuous.

Theorem 2. If Y is a locally compact Hausdorff space and X and Z are spaces, a map g: $Z \to X^Y$ is continuous if and only if $E \circ (g \times 1_Y)$: $Z \times Y \to X$ is continuous.

Definition 4. A Hausdorff space is para-compact if each open covering has an open nbd-finite refinement.

Theorem 3 ([2], VIII, 2.2). If X is para-compact then X is normal.

We finish with a summary of some results about cozero sets. These results are discussed in detail by [2] (XX, Section 3).

Definition 5. Let X be a space. An open set $U \subset X$ is called a cozero set if there is a continuous c: $X \to I (I = [0,1])$ such that $c^{-1}(0) = X - U$. c is called the characterizing function for U.

Theorem 4. Let X be a space. Then

(a) The intersection of finitely many cozero sets is a cozero set.

(b) The union of any nbd-finite family of cozero sets is a cozero set.

(c) Let U be a cozero set in X and let $K \subset I$ be compact. Then, the set (K,U) is a cozero set.

Theorem 5. Let X be a space, and let $\{U_\beta\}$ be a covering of X by cozero sets. Assume that $\{U_\beta\}$ can be decomposed into a countable collection $\{U^k_{\beta_k}\}$ of nbd-finite families. Then $\{U_\beta\}$ has a nbd-finite refinement by cozero sets.

CHAPTER I

THE EXTENSION AND LIFTING PROBLEMS

Let X and Y be topological spaces and let A be a subset of X. Suppose there is a map (continuous function) $f\colon A \to Y$. The extension problem for f in this case would be to find a map $F\colon X \to Y$ such that $F(a) = f(a)$ for all $a \in A$.

In a more general setting; given two maps $f\colon X \to Y$ and $p\colon X \to Z$, the generalized extension problem for f is to find a map $g\colon Z \to Y$ such that $g \circ p = f$. In particular then, if $f\colon A \to Y$ is a map and $p\colon A \subseteq X$ denotes the inclusion map, the problem of finding F above is the same as finding a map $g\colon X \to Y$ such that $g \circ p = f$.

For the mappings f and p above, the existence of g is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ Z \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Another well-known problem, which is dual to the generalized extension problem, is the lifting problem.

Let f be a mapping from X onto Y. A <u>section</u> s of f is a mapping from Y into X such that s is a right inverse of f, i.e. $f \circ s = 1_Y$.

Given maps $f\colon X \to Y$ and $p\colon \tilde{X} \to Y$, a <u>lifting</u> of f is a map f' represented by the dotted arrow which makes the following diagram commutative.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ X \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & Y \end{tikzcd}$$

If f is the identity map 1_Y from Y onto Y, then the lifting f' (if such a lifting exists) is a section of p.

Extensions and liftings do not always exist; for example:

Let S^1 be the unit sphere and let E^1 be the unit ball in R_2 (Euclidean plane).

If $1_{S^1}\colon S^1 \to S^1$ is the identity map, then there is no map $F\colon E^1 \to S^1$ so that $F(x) = 1_{S^1}(x) = x$ for $x \in S^1$ (this fact is equivalent to the Brouwer fixed point theorem for E^1).

Also, if $p\colon R_1 \to S^1$ is the exponential map $p(x) = e^{ix}$ then the identity map can not be lifted in R, because any map $g\colon S^1 \to R_1$ must send at least one pair of points to a single point.

1. The Extension Problem

Let us consider again the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ Z \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

where f and p are given maps.

It is clear from this diagram that if we hope to find a continuous g such that gp = f, it is necessary that if $x,x'\varepsilon p^{-1}(z)$ for some $z\varepsilon Z$ then $f(x) = f(x')$. $p^{-1}(z)$ is called the fiber of p over z. Thus a necessary condition for the existence of g is that f be constant on the fibers of p.

Proposition 1. Given maps $f\colon X\to Y$ and $p\colon X\to Z$, a function $g\colon Z\to Y$ with gp = f exists if and only if f is constant on the fibers of p.

Proof. The necessity has already been established above.

If if is constant on the fibers of p, simply define $g\colon Z\to Y$ by $g(z) = y$ if $y = f(x)$ where $x\varepsilon p^{-1}(z)$. Then g is well defined and gp = f.

Throughout the remainder of this section $f\colon X\to Y$ and $p\colon X\to Z$ will be mappings such that f is constant on the fibers of p. Then the generalized extension problem would be solved if the map p had a section s.

Finding such a section is equivalent to the problem of finding a subset A of X for which p/A: $A \to Z$ is a surjection with a continuous inverse.

Definition 1. A mapping f: $X \to Y$ is said to be compact if for every compact subset C of Y, $f^{-1}(C)$ is also compact. This is not the usual definition of a compact map $[f^{-1}(C)]$ compact for compact $C \subset f(X)$.

Proof. Let C be a closed set in X and let y_o be a limit point of $f(C)$.

Since Y is first countable, there is a sequence $\{y_n\}$ in $f(C)$ converging to y_o. Clearly, $\{y_n\} \cup \{y\}$ is a compact subset of Y. Hence, $f^{-1}(\{y_n\} \cup \{y\})$ is compact in X.

Let $x_n \varepsilon C \cap f^{-1}(y_n)$. Then $\{x_n\}$ is a sequence in $f^{-1}(\{y_n\} \cup \{y\})$. Therefore $\{x_n\}$ has a limit point, say x_o. Since C is closed, $x_o \varepsilon C$.

By the continuity of f, it follows that $f(x_o)$ is a limit point of $\{y_n\}$. Since Y is Hausdorff it follows that $f(x_o) = y_o$, and hence that $y_o \varepsilon f(C)$.

Corollary 2.1 With the hypothesis of Propositions 1 and 2 it follows that, if f/A: $A \to Y$ is a bijection and A is closed, then f has a section.

A similar result to Proposition 2 may be obtained by using nets.

Proposition 3. Let f: $X \to Y$ be a continuous function. Assume that Y is a locally compact Hausdorff space. Then f is closed if f is a compact map.

Proof. Let C be a closed subset of X, and let y_o be a limit point of f(C).

Let K be a compact neighborhood of y_o and let $\{y_\alpha\}$ be a net in $K \cap f(C)$ converging to y_o.

Let $x_\alpha \varepsilon f^{-1}(y_\alpha) \cap C$. Since $f^{-1}(K)$ is compact in X it follows that the net $\{x_\alpha\}$ has a limit point, say x_o. Since C is closed, $x_o \varepsilon C$.

Therefore, since f is continuous, $f(x_o)$ is also a limit point of $\{y_\alpha\}$. Since Y is Hausdorff, $f(x_o) = y_o = y_o \varepsilon f(C)$.

Corollary 3.1 With the hypothesis of Proposition 2 it follows that if f/A: A→Y is a bijection and A is closed, then f has a section.

Proposition 4. Let f: X→Y and p: X→Z be continuous. Then if p,f,X and Z satisfy the hypothesis in Corollary 2.1 or Corollary 3.1, it follows that there is a mapping g: Z→Y such that f = gp.

Proof. From the corollaries it follows that p has a Section s: Z→X. Define g = fs.

Let p: X→Z be a surjective map and let R be the relation on X defined by: xRX' if p(x) = p(x'). This relation is an equivalence relation. Let [x] denote the equivalence class containing x.

Let X/R be the space of equivalence classes with the identification or quotient topology induced by the function h: X→X/R defined by: h(x) = [x] (for a discussion on identification topologies see [2], p. 121).

Lemma 1. Let $f: X \to Z$ be a continuous surjection. Then f is an identification if and only if for each function $g: Z \to Y$, the continuity of $g \circ f$ implies that of g.

Proof. Necessity: Assume that f is an identification and that $g \circ f$ is continuous. Then, for each open $U \subset Y$, $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. Since f is an identification, it follows that $g^{-1}(U)$ is open in Z: g is therefore continuous.

Sufficiency: Let X/R be Z with the identification topology induced by f. Let $f': X \to X/R$ be the identification map and let $i: Z \to X/R$ be defined by $i(z) = [x]$, $x \in f^{-1}(z)$. Then $f' = if$. By the condition, it then follows that i is continuous. But i^{-1} is clearly continuous. Therefore i is a homeomorphism and hence, f is an identification.

Combining Proposition 1 with Lemma 1 we obtain

Proposition 5. Let p be a surjective map and let F be the collection of maps $f: X \to Y$ for which f is constant on the fibers of p. There is a solution to the generalized extension problem for $f \in F$ if and only if the map p is an identification.

Proof. By Proposition 1 a necessary and sufficient condition for the existence of a function $g: Z \to Y$ is that f be constant on the fibers of p. The proposition now follows from Lemma 1.

2. The Lifting Problem and Lifting Functions

In this section we shall concentrate on the lifting problem.

Proposition 1. Let $f\colon X \to Y$ and $p\colon \tilde{X} \to Y$ be mappings. If p, Y and $\tilde{X}$ satisfy the hypothesis in Corollary 2.1 or Corollary 3.1, Section 1 (on f, X and Y), then there is a lifting f' of f.

Proof. If s is a section of p, define $f' = sf$.

Proposition 2. Let A and B be topological spaces and let $\tilde{X} = A \times B$. Let $p\colon \tilde{X} \to A$ be the projection onto the first factor. Then, given any map $f\colon X \to A$ there exists a map $f'\colon X \to \tilde{X}$ such that $pf' = f$.

Proof. If $f\colon X \to A$ is a map, define $f'\colon X \to \tilde{X}$ by $f'(x) = (f(x), b_o)$ for some fixed $b_o \in B$. Clearly, f' is continuous and $pf' = f$.

Proposition 3. If $p\colon \tilde{X} \to Y$ and $f\colon X \to Y$ are maps such that $f^{-1}(y)$ is open for every $y \in Y$, then there is a lifting f' of f.

Proof. Since $\{f^{-1}(y)\}_{y \in Y}$ covers X and $f^{-1}(y_1) \cap f^{-1}(y_2) = \phi$ if $y_1 \neq y_2$, it follows that the function $f'\colon X \to \tilde{X}$, defined by $f'(x) = \tilde{x} \in p^{-1}(f(x))$, $[f'$ constant on $f^{-1}(f(x))]$, is continuous.

Definition 1. Let $p\colon X \to Y$ be a map. An open set $U \subset Y$ is said to be <u>evenly covered</u> by p if $p^{-1}(U)$ is the union of a family of pairwise disjoint open sets such that each one of these open sets is homeomorphic to U under p.

Definition 2. Given a map $p\colon X \to Y$, we say that p is a <u>covering projection</u> if for every element $y \in Y$ there exists an open set U in Y containing y and which is evenly covered by p.

Examples: A homemorphism is a covering projection.

The exponential map $\exp\colon R_1 \to S^1$ defined by $\exp(x) = e^{2\pi i x}$ is a covering projection.

Also the map $p_n\colon S^1 \to S^1$ defined by $p_n(z) = z^n$ for any positive integer n is a covering projection.

Definition 3. Let X be a topological space. A path ω from a point x_1 to a point x_2 in X is a continuous function from the unit interval into X such that $\omega(0) = x_1$ and $\omega(1) = x_2$.

Proposition 3. Let $p\colon \tilde{X} \to Y$ be a covering projection and let $\omega\colon I \to Y$ be a path in Y such that $\omega(0) = p(\tilde{x}_o)$ for some $\tilde{x}_o \in \tilde{X}$. Then, there exists a path $\tilde{\omega}$ in $\tilde{X}$ such that $p\tilde{\omega} = \omega$ and $\tilde{\omega}(0) = \tilde{x}_o$.

Proof. Let $\omega\colon I \to Y$ be a path. If $t \in I$ then there exists an open set U_t in Y which contains $\omega(t)$ and which is evenly covered by p.

Since ω is continuous there exists a $\delta_t > 0$ such that if $t' \in (t-\delta_t, t+\delta_t)$ then $\omega(t') \in U_t$.

Since p is a covering projection, there is for each $t \in I$ an open set U_t in Y containing $\omega(t)$ and a $\delta_t > 0$ such that the conditions above are satisfied.

The family $\{(t-\delta_t, t+\delta_t) | t \in I\}$ is an open covering of the compact set I. Therefore there are $t_1, \cdots, t_n$ in I such that $\bigcup_{i=1}^{n} (t_i-\delta_i, t_i+\delta_i) \supset I$. We may assume that $t_1 < t_2 < \cdots < t_n$ and that there are no superfluous subintervals in the finite subcovering. Clearly $\omega(0) \in [0, t_1+\delta_1)$ and $(t_{i-1}-\delta_{i-1}, t_{i-1}+\delta_{i-1}) \cap (t_i-\delta_i, t_i+\delta_i) \neq \phi$.

Let $\tilde{x}_o$ be an element in $\tilde{X}$ such that $p(\tilde{x}_o) = \omega(0)$.

Since p is a covering projection, $p^{-1}(U_1) = \cup F_1$ where F_1 is a family of disjoint open sets such that if $V \in F_1$ then $p/V\colon V \to U_1$ is a homeomorphism.

Clearly, there is a unique $V_1 \in F_1$ such that $\tilde{x}_o \in V_1$.

Let $\bar{t}_1 \in [0,t_1+\delta_1) \cap (t_2-\delta_2,t_2+\delta_2)$. Then $\omega(\bar{t}_1) \in U_1 \cap U_2$. Let $\tilde{x}_1$ be that unique element in V_1 such that $p(\tilde{x}_1) = \omega(\bar{t}_1)$.

Define $\tilde{\omega}_1$: $[0,\bar{t}_1] \to \tilde{X}$ by $\tilde{\omega}_1(t') = ((p/V_1)^{-1} \circ \omega)(t')$ for $t' \in [0,\bar{t}_1]$. Then, $\tilde{\omega}_1(0) = \tilde{x}_0$ and $\tilde{\omega}_1(\bar{t}_1) = \tilde{x}_1$. Again, since p is a covering projection, $p^{-1}(U_2) = UF_2$ where F_2 is a family of disjoint open sets such that if $V \in F_2$ then p/V: $V \to U_2$ is a homeomorphism.

Now, $p(\tilde{x}_1) = \omega(\bar{t}_1) \in U_1 \cap U_2$. Let V_2 be that unique open set in F_2 such that $\tilde{x}_1 \in V_2$.

Let $\bar{t}_2 \in (t_2-\delta_2,t_2+\delta_2) \cap (t_3-\delta_3,t_3+\delta_3)$.

Then $\omega(\bar{t}_2) \in U_2 \cap U_3$. Let $\tilde{x}_2$ be that unique element in V_2 such that $p(\tilde{x}_2) = \omega(\bar{t}_2)$.

Define $\tilde{\omega}_2$: $[\bar{t}_1,\bar{t}_2] \to X$ by $\tilde{\omega}_2(t') = ((p/V_2)^{-1} \circ \omega)(t')$ for $t' \in [\bar{t}_1,\bar{t}_2]$. Then $\omega_2(\bar{t}_1) = \tilde{x}_1$, and $\tilde{\omega}_2(\bar{t}_2) = \tilde{x}_2$. Continue this process n times. We then have the following: $\bar{t}_1,\bar{t}_2,\ldots,\bar{t}_{n-1}$ such that $\omega(\bar{t}_i) \in U_i \cap U_{i+1}$; $\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_{n-1}$ in $\tilde{X}$ such that $p(\tilde{x}_i) = \omega(\bar{t}_i)$; open sets $V_1,\ldots,V_n$ such that $\tilde{x}_{i-1},\tilde{x}_i \in V_i$ and p/V_i: $V_i \to U_i$ is a homeomorphism and finally, we have $\tilde{\omega}_1,\ldots,\tilde{\omega}_n$ where $\tilde{\omega}_i = [\bar{t}_{i-1},\bar{t}_i] \to \tilde{X}$ is the continuous function given by $\tilde{\omega}_j(t') = \big((p/V_i)^{-1} \circ \omega\big)(t')$ for $t' \in [\bar{t}_{i-1},\bar{t}_i]$, such that $\tilde{\omega}_i(\bar{t}_{i-1}) = \tilde{x}_{i-1}$ and $\tilde{\omega}_i(\bar{t}_i) = \tilde{x}_i$. Here, $\bar{t}_n = 1$ and $\bar{t}_o = 0$.

Define $\tilde{\omega}(t) = \tilde{\omega}_i(t)$ if $t \in [\bar{t}_{i-1},\bar{t}_j]$.

This path $\tilde{\omega}$ is then the desired lifting for ω.

It may seem that the construction above depends upon the choice of U's. However, once we specify a single point of $\tilde{\omega}$, then the rest of the path is uniquely determined. More precisely:

Proposition 4. Let p: $\tilde{X} \to Y$ be a covering projection and let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ be paths in $\tilde{X}$ such that $p\tilde{\omega}_1 = p\tilde{\omega}_2$ and $\tilde{\omega}_1(t_o) = \tilde{\omega}_2(t_o)$ for some $t_o \varepsilon I$. Then $\tilde{\omega}_1(t) = \tilde{\omega}_2(t)$ for all $t \varepsilon I$.

Proof. Let $A = \{t | \tilde{\omega}_1(t) = \tilde{\omega}_2(t)\}$. By hypothesis $t_o \varepsilon A$. It will be proved that A is open and closed in I. This implies, since I is connected, that A = I.

Let $t_1 \varepsilon A$ and let U be a neighborhood of $p\tilde{\omega}_1(t_1)$ evenly covered by p.

Let F be a family of disjoint open sets in $\tilde{X}$ such that $\cup F = p^{-1}(U)$ and such that if $V \varepsilon F$ then $p/V: V \to U$ is a homeomorphism.

Let V be that unique open set in F such that $\tilde{\omega}_1(t_1) = \tilde{\omega}_2(t_1) \varepsilon V$.

Let $t \varepsilon \tilde{\omega}_1^{-1}(V) \cap \tilde{\omega}_2^{-1}(V)$. Then $\tilde{\omega}_1(t) \varepsilon V$ and $\omega_2(t) \varepsilon V$. Also $p\tilde{\omega}_1(t) = p\tilde{\omega}_2(t)$ by hypothesis. But p is one to one on V. Thus $\tilde{\omega}_1(t) = \tilde{\omega}_2(t)$.

It follows that $\tilde{\omega}_1^{-1}(V) \cap \tilde{\omega}_2^{-1}(V) \subset A$. Therefore, A is open in I.

Let t_1 be a limit point of A. Let U be the same open set above and let V_1 and V_2 be open sets in $\tilde{X}$ containing $\tilde{\omega}_1(t_1)$ and $\tilde{\omega}_2(t_1)$, respectively, where $V_1, V_2 \varepsilon F$ (F defined as above).

If $V_1 = V_2$ it follows that $\tilde{\omega}_1(t_1) = \tilde{\omega}_2(t_1)$ and hence, that $t_1 \varepsilon A$.

If $V_1 \neq V_2$ then $V_1 \cap V_2 = \phi$. But this is impossible since $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are continuous and t_1 is a limit point of A.

Thus $t_1 \in A \Rightarrow A$ is closed in I.

Definition 4. A space X is said to be <u>contractible</u> to x_o in X if there is a continuous function $F: X \times I \to X$ such that $F(x,0) = x_o$ and $F(x,1) = x$ for all $x \in X$.

Example. The unit ball E^n in R^n is contractible to the origin. Let $F: E^n \times I \to E^n$ be given by $F(x,t) = t \cdot x$ for $x \in E^n$.

Proposition 5. Let $p: (\tilde{X},\tilde{x}_o) \to (Y,y_o)$ be a covering projection and let $f: (X,x_o) \to (Y,y_o)$ be a map. If X is contractible to x_o, then f can be lifted to $(\tilde{X},\tilde{x}_o)$.

Proof. Since X is contractible to x_o there exists a map $F: X \times I \to Y$ such that $F(x,0) = y_o$ and $F(x,1) = f(x)$ for all $x \in X$. Let $\omega_{f(x)}: I \to Y$ be defined by $\omega_{f(x)}(t) = F(x,t)$. Then $\omega_{f(x)}(1) = f(x)$ and $\omega_{f(x)}(0) = y_o$. Define $f': (X,x_o) \to (\tilde{X},\tilde{x}_o)$ by $f'(x) = \tilde{\omega}_{f(x)}(1)$ where $\omega_{f(x)}$ is the unique lifting of $\omega_{f(x)}$ beginning at $\tilde{x}_o$.

Recall the construction of $\tilde{\omega}_{f(x)}$ from Proposition 3. Using the same notation as in the proof of Proposition 3, U_n is an evenly covered neighborhood of $f(x)$.

Let $x_1 \in X$ and let O be an open set in X such that $f'(x_1) \in O$.

Since p is an open map, it follows that $p(O)$ is an open set.

If $U_1,\ldots,U_n$ are the construction sets for $\tilde{\omega}_{f(x_1)}$ we have that $F(x_1,[\bar{t}_{i-1},\bar{t}_i]) \subset U_2$ for $i=1,\ldots,n$.

Since $[\bar{t}_{i-1},\bar{t}_i]$ is compact, there exists an open set S_i in X which contains x_1 and such that $F(S_1,[\bar{t}_{i-1},\bar{t}_i]) \subset U_i$.

Using the notation of Proposition 3, V_n is the open set in X containing $f'(x_1)$ such that $p/V_n\colon V_n \approx U_n$.

Let $V = V_n \cap O$ then $f'(x_1) \varepsilon V$ and $p/V\colon V \approx p(V) \subset U_n$.

Let N be an open set in X such that $x_1 \varepsilon N$ and $F(N,1) \subset p(V) \subset U_n$.

Let $T = (\bigcap_{i=1}^{n} S_i) \cap N$. Then $x_1 \varepsilon T$.

Suppose $x \varepsilon T$. Then $F(x_p[\bar{t}_{i-1},\bar{t}_i]) \subset U_i$ for $i=1,\ldots,n$. As in Proposition 3, we may construct a path $\tilde{\omega}$ such that $\tilde{\omega}(0) = \tilde{x}_o$, $\tilde{\omega}([\bar{t}_{i-1},\bar{t}_i]) \subset V_i$ for $i=1,\ldots,n$, and $p\tilde{\omega} = \tilde{\omega}_{f(x)}$.

Thus $p\tilde{\omega}(1) \varepsilon p(V)$ and since $p/V_n\colon V_n \approx U_n$, it follows that $\tilde{\omega}(1) \varepsilon V$.

But by Proposition 4 we know that $\tilde{\omega} = \tilde{\omega}_{f(x)}$. Since $f'(x) = \tilde{\omega}_{f(x)}(1) = \tilde{\omega}(1) \varepsilon V \subset O$ and x is arbitrary in T, we have $f'(T) \subset O$ and hence, f' is continuous.

Definition 5. Let ω_1 and ω_2 be paths in X. We say that ω_1 and ω_2 are <u>homotopic paths</u>, and we write $\omega_1 \simeq \omega_2$, if there exists a map $F\colon I \times I \to X$ such that $F(t,o) = \omega_1(t)$, $F(t,1) = \omega_2(t)$ for all $t \varepsilon I$; and $F(0,t') = \omega_1(0)$, $F(1,t') = \omega_1(1)$ for all $t' \varepsilon I$.

Thus a necessary condition for ω_1 and ω_2 to be homotopic is that $\omega_1(0) = \omega_2(0)$ and $\omega_2(1) = \omega_1(1)$.

Definition 6. If ω_1 and ω_2 are paths in X such that $\omega_1(1) = \omega_2(0)$, then $\omega_1 * \omega_2$ is the path defined by

$$(\omega_1 * \omega_2)(t) = \begin{cases} \omega_1(2t) & \text{if } 0 \le t \le 1/2 \\ \omega_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

It is easy to show that $\omega_1 * \omega_2$ is continuous.

Let p: $\tilde{X} \to Y$ be a map. Define $\bar{Y}$ to be the subspace of $\tilde{X} \times Y^I$ given by

$$\bar{Y} = \{(\tilde{x},\omega) \mid (\tilde{x},\omega) \in \tilde{X} \times Y^I \text{ and } p(\tilde{x}) = \omega(0)\}$$

Define $\bar{p}$: $\tilde{X}^I \to \bar{Y}$ to be the map $\bar{p}(\tilde{\omega}) = (\tilde{\omega}(0), p\tilde{\omega})$ where $\tilde{X}^I$ and Y^I are given the compact open topology described in Chapter 0. Then it is clear that $\bar{p}$ is a map.

Definition 7. Given a map p: $\tilde{X} \to Y$, a <u>lifting function</u> for p is a continuous function λ: $\bar{Y} \to \tilde{X}^I$ such that $\bar{p}\lambda = 1_{\bar{Y}}$.

Thus a lifting function assigns to each point $\tilde{x} \in \tilde{X}$ and path ω in Y starting at $p(\tilde{x})$, a path $\lambda(\tilde{x},\omega)$ in $\tilde{X}$ starting at $\tilde{x}$ such that $p\lambda(\tilde{x},\omega) = \omega$.

Definition 8. A map p: $\tilde{X} \to Y$ is said to have the <u>unique path lifting</u> property if given paths $\tilde{\omega}_1$ and $\tilde{\omega}_2$ in $\tilde{X}$ such that $\tilde{\omega}_1(0) = \tilde{\omega}_2(0)$ and $p\tilde{\omega}_1 = p\tilde{\omega}_2$, then $\tilde{\omega}_1 = \tilde{\omega}_2$.

As Proposition 4 shows, if p is a covering projection, then it has the unique path lifting property.

Proposition 6. If p: $\tilde{X} \to Y$ is a covering projection then p has a lifting function.

Proof. Let $(\tilde{x},\omega) \in \bar{Y}$. Define $\lambda(x,\omega)$ to be the path $\tilde{\omega}$ starting at $\tilde{x}$ which was constructed in the proof of Proposition 3. Let $U_1,\ldots,U_n$ and $V_1,\ldots,V_n$ be the sets used to construct $\tilde{\omega}$ in the proof of

Proposition 3. Recall that $\tilde{\omega}([\bar{t}_{i-1},\bar{t}_i]) \subset V_i$ and $\omega([\bar{t}_{i-1},\bar{t}_i]) \subset U_i$ for $i=1,\ldots,n$.

By Proposition 4 we see that λ is a function from $\bar{Y}$ to $\tilde{X}^I$. We show next that λ is continuous.

Let (K,V) be a subbasic open set in $\tilde{X}^I$ containing $\tilde{\omega} = \lambda(\tilde{x},\omega)$. By definition $\tilde{\omega}(0) = \tilde{x} \in V_1$ and clearly $\tilde{\omega}([\bar{t}_{i-1},\bar{t}_i] \cap K) \subset V \cap V_i$.

Let $S = \left(\bigcap_{i=1}^{n} ([\bar{t}_{i-1},\bar{t}_i],U_i)\right) \cap \left(\bigcap_{i=1}^{n} (K \cap [\bar{t}_{i-1},\bar{t}_i],p(V_i \cap V))\right)$. Then, $\omega \in S$. If $\tilde{x} \in V$ then let $U = V \cap V_1$; otherwise let $U = V_1$. Then $(\tilde{x},\omega) \in U \times S$.

Let $(\tilde{x}_1,\omega_1) \in U \times S \cap \bar{Y}$. Then, as in the proof of Proposition 3. a path $\tilde{\omega}_1$ in $\tilde{X}$ may be constructed such that $\tilde{\omega}_1(0) = \tilde{x}_1$, $p\tilde{\omega}_1 = \tilde{\omega}_1$ and $\tilde{\omega}_1([\bar{t}_{i-1},\bar{t}_i]) \subset V_i$ for $i=1,\ldots,n$. Since p is a homeomorphism on V_i and $\omega_1 \in \bigcap_{i=1}^{n} \left(K \cap [\bar{t}_{i-1},\bar{t}_i],p(V_i \cap V)\right)$ it follows that $\tilde{\omega}_1 \in (K,V)$. But $\tilde{\omega}_1(0) = \tilde{x}_1$ and hence, by Proposition 4, it follows that $\tilde{\omega}_1 = \lambda(\tilde{x}_1,\omega_1)$.

We have shown that $\lambda(U \times S \cap \bar{Y}) \subset (K,V)$. Therefore λ is continuous.

Definition 9. A space X is said to be <u>locally path connected</u> if it has a basis consisting of path connected open sets.

It follows immediately that:

Proposition 7. A space is locally path connected if and only if path components of open sets are open.

Lemma 1. Let $p\colon \tilde{X} \to Y$ be a map and suppose that p has a lifting function λ. Assume that λ lifts constant paths in $p(\tilde{X})$ to constant paths in $\tilde{X}$. Then, if ω_1 and ω_2 are paths in Y such that $\omega_1 \simeq \omega_2$ and

$p(\tilde{x}_o) = \omega_1(0) = \omega_2(0)$ for some $\tilde{x}_o \in \tilde{X}$, it follows that $\lambda(\tilde{x}_o,\omega_1) \simeq \lambda(\tilde{x}_o,\omega_2)$.

Proof. Let $F\colon I \times I \to Y$ be a homotopy from ω_1 to ω_2. Then $F(t,0) = \omega_1(t)$, $F(t,1) = \omega_2(t)$ and $F(0,t') = \omega_1(0)$, $F(1,t') = \omega_1(1)$ for all $t' \in I$.

For each $t \in I$ define $F_t\colon I \to Y$ by $F_t(t') = F(t',t)$. Then $F_o = \omega_1$ and $F_1 = \omega_2$.

Next define $f\colon I \to \bar{Y}$ to be the map $f(t) = (\tilde{x}_o,F_t)$. Clearly, $f(t) \in \bar{Y}$. We shall prove that f is continuous.

Let $t_o \in I$ and let (K,V) be a subbasic open set in Y^I containing F_{t_0}.

If $t' \in K$ then $F_{t_0}(t') = F(t',t_0) \in V$.

Since F is continuous there exists open sets $U_{t'}$ and $0_{t'}$ in I such that $t' \in U_{t'}$, $t_0 \in 0_{t'}$ and $F(U_{t'},0_{t'}) \subset V$.

Since for each $t' \in K$ there are corresponding $U_{t'}$ and $0_{t'}$, we have an open covering $\{U_{t'}\}$ of K. Since K is compact there are $U_{t_1},\ldots,U_{t_n}$ such that $K \subset \bigcup_{i=1}^{n} U_{t_n}$.

Let $0 = \bigcap_{i=1}^{n} 0_{t_i}$ where 0_{t_i} corresponds to the U_{t_i} above. Then $t_0 \in 0$.

Now, let $t'' \in 0$ and let $t' \in K$. Thus $t' \in U_{t_i}$ for some i. Clearly $t'' \in 0_{t_2}$. Therefore $F_{t''}(t') = F(t',t'') \in V$. Thus $F_{t''} \in (K,V)$ and it follows $f(0) \subset (K,V)$. $\therefore$ f is continuous.

Define $G\colon I\times I \to X$ as the composition $I\times I \xrightarrow{1_I\times f} I\times\bar{Y} \xrightarrow{1_I\times\lambda} I\times\tilde{X}^I \xrightarrow{E} \tilde{X}$ where E is the evaluation map defined in Chapter 0.

Since f is continuous, it follows from Theorem 2 in Chapter 0 that G is a continuous function.

Now,

$$G(t,0) = (E\circ(1_I\times\lambda) \circ (1_I\times f))(t,0)$$

$$= (E\circ(1_I\times\lambda))(t,f(0))$$

$$= E(t,\lambda(\tilde{x}_0,F_0)) = \lambda(\tilde{x}_0,F_0)(t)$$

$$= \lambda(\tilde{x}_0,\omega_1)(t)$$

and

$$G(t,1) = \lambda(\tilde{x}_0,F_1)(t) = \lambda(\tilde{x}_0,\omega_2)(t).$$

Also,

$$G(0,t) = \lambda(\tilde{x}_0,F_t)(0) = \tilde{x}_0 \quad \text{for all } t \in I$$

and

$$pG(1,t) = p\lambda(\tilde{x}_0,F_t)(1) = F_t(1) = \omega_1(1)$$

for all $t \in I$. Thus $pG(1,t)$ is a constant path and therefore the hypothesis implies that $G(1,t)$ is the same point in $\tilde{X}$ for all $t \in I$. We have shown that G is a homotopy from $\lambda(\tilde{x}_0,\omega_1)$ to $\lambda(\tilde{x}_0,\omega_2)$.

Lemma 2. Let λ be a lifting function for $p\colon \tilde{X}\to Y$. If p has the unique path lifting property then λ lifts constant paths in $p(\tilde{X})$ to constant paths in $\tilde{X}$.

Proof. The constant path at $\tilde{x}$ clearly must be the lifting of the constant path at $p(\tilde{x})$.

Definition 10. Given a path ω, ω^{-1} is defined by $\omega^{-1}(t) = \omega(1-t)$ for $t \in I$.

We are now ready to prove the main result of this section.

Proposition 7. Let $p\colon (\tilde{X},\tilde{x}_0) \to (Y,y_0)$ and $f\colon (X,x_0) \to (Y,y_0)$ be maps. Assume that X is path connected and locally path connected. If p has the unique path lifting property and if it has a lifting function λ, then a lifting $f'\colon (X,x_0) \to (\tilde{X},\tilde{x}_0)$ for f exists if and only if, given a closed path ω at x_0 in X, there exists a closed path $\tilde{\omega}$ at $\tilde{x}_0$ in $\tilde{X}$ such that $f\omega \simeq p\tilde{\omega}$.

Proof. The condition is clearly necessary. We show that the converse is also true.

Let $x \in X$ and let ω_x be a path from x_0 to x. Let us fix this path once and for all. Then $f\omega_x$ is a path from y_0 to $f(x)$.

Define $f'\colon (X,x_0) \to (\tilde{X},\tilde{x}_0)$ by $f'(x) = \lambda(\tilde{x}_0,f\omega_x)(1)$.

Clearly $pf'(x) = f(x)$. We show next that f' is continuous.

Let $x \in X$ and let O be an open set in $\tilde{X}$ containing $f'(x) = \lambda(\tilde{x}_0,f\omega_x)(1)$.

Since λ is continuous there is an open set $\bigcap_{i=1}^{n} (K_i,U_i)$ in Y^I containing $f\omega_x$ such that $\lambda([\{\tilde{x}_0\} \times \bigcap_{i=1}^{n} (K_i,U_i)] \cap \bar{Y}) \subset (\{1\},O)$.

Enumerate the K_i's so that for some integer k, $0 \leq k \leq n$, $1 \in K_1 \cap K_2 \cap \ldots \cap K_k$ and $1 \notin K_{k+1} \cup \ldots \cup K_n$. Then $f\omega_x(1) \in \bigcap_{i=1}^{k} U_i$ and $\omega_x(1) \in f^{-1}(\bigcap_{i=1}^{k} U_i)$.

Since X is locally path connected, the path component V in $f^{-1}(\bigcap_{i=1}^{k} U_i)$ containing $\omega_x(1) = x$ is an open set in X.

Choose $t_0 < 1$ so that $\omega_x([t_0,1]) \subset V$ and $[t_0,1] \cap (\bigcup_{i=K+1}^{n} K_i) = \phi$.

We now show that $f'(V) \subset 0$ from which it will follow that f' is continuous.

Let $x' \in V$ and let ω' be a path in V from $\omega_x(t_0)$ to x'. Define $\bar{\omega}$: $I \to X$ by

$$\bar{\omega}(t) = \begin{cases} \omega_x(t) & \text{if } 0 \leq t \leq t_0 \\ \omega'\left(\dfrac{t-t_0}{1-t_0}\right) & \text{if } t_0 \leq t \leq 1. \end{cases}$$

Then $\bar{\omega}$ is a path from x_0 to x'. For $i > k$, $f\bar{\omega}(K_i) = f\omega_x(K_i) \subset U_i$. For $i \leq k$,

$$\begin{aligned} f\bar{\omega}(K_i) &= f\bar{\omega}(K_i \cap [0,t_0]) \cup f\bar{\omega}(K_i \cap [t_0,1]) \\ &\subset f\omega_x(K_i) \cup f\omega'(I) \\ &\subset U_i \cup f(V) = U_i. \end{aligned}$$

We have shown that $f\bar{\omega} \in \bigcap_{i=1}^{n} (K_i,U_i)$.

Thus $\lambda(\tilde{x}_0, f\bar{\omega}) \in (\{1\},0)$ and hence, $\bar{x} = \lambda(\tilde{x}_0, f\bar{\omega})(1) \in 0$. We show that $f^1(x') = \bar{x}$ and the proof will then be completed.

Consider the path $\omega_{x'}$ in X from x_0 to x' used to determine $f^1(x')$, and let $\omega = \bar{\omega} * \omega_{x'}^{-1}$. Then $\omega(0) = \bar{\omega}(0) = x_0$ and $\omega(1) = \omega_{x'}^{-1}(2-1) = \omega_{x'}^{-1}(1) = \omega_{x'}(0) = x_0$. Therefore ω is closed at x_0 and by hypothesis, there exists a closed path $\tilde{\omega}$ at $\tilde{x}_0$ in $\tilde{X}$ such that $p\tilde{\omega} \simeq f\omega$.

Clearly,

$$\begin{aligned} f\omega &= f(\bar{\omega}*\omega_{x'}^{-1}) \\ &= f\bar{\omega}*f\omega_{x'}^{-1} = f\bar{\omega}*(f\omega_{x'})^{-1}. \end{aligned}$$

By unique path lifting $\lambda(\tilde{x}_0, p\tilde{\omega}) = \tilde{\omega}$.

It then follows from Lemmas 1 and 2 that $\tilde{\omega} = \lambda(\tilde{x}_0, p\tilde{\omega}) \simeq \lambda(\tilde{x}_0, f\omega)$ and therefore $\lambda(\tilde{x}_0, f\omega)(1) = \tilde{\omega}(1) = \tilde{x}_0$. Now,

$$\begin{aligned} p\lambda(\tilde{x}_0, f\omega)\left(\frac{t}{2} + \frac{1}{2}\right) &= f\omega\left(\frac{t}{2} + \frac{1}{2}\right) \\ &= f\omega_{x'}^{-1}\left(2\left(\frac{t}{2} + \frac{1}{2}\right) - 1\right) \\ &= f\omega_{x'}^{-1}(t) \end{aligned}$$

By unique path lifting then,

$$\lambda(\tilde{x}_0, f\omega)\left(\frac{t}{2} + \frac{1}{2}\right) = \lambda\left(\bar{x}, (f\omega_{x'})^{-1}\right)(t).$$

Thus $\lambda\left(\bar{x},(f\omega_{x'})^{-1}\right)(1) = x_0$. Therefore

$$[\lambda\left(\bar{x},(f\omega_{x'})^{-1}\right)]^{-1}(0) = \lambda\left(\bar{x},(f\omega_{x'})^{-1}\right)(1) = \tilde{x}_0.$$

Since $p\lambda(\bar{x},(f\omega_{x'})^{-1}) = (f\omega_{x'})^{-1}$ it follows that

$$\begin{aligned} p[\lambda(\bar{x},(f\omega_{x'})^{-1}]^{-1} &= [p\lambda(\bar{x},(f\omega_{x'})^{-1})]^{-1} \\ &= \left((f\omega_{x'})^{-1}\right)^{-1} \\ &= f\omega_{x'}. \end{aligned}$$

By unique path lifting

$$[\lambda(\bar{x},(f\omega_{x'})^{-1})]^{-1} = \lambda(\tilde{x}_0,f\omega_{x'})$$

and therefore

$$\bar{x} = \lambda(\bar{x},(f\omega_{x'})^{-1})(0) = [\lambda(\bar{x},(f\omega_{x'})^{-1})]^{-1}(1) = \lambda(\tilde{x}_0,f\omega_{x'})(1).$$

But, by definition, $\lambda(\tilde{x}_0,f\omega_{x'})(1) = f'(x')$. We have shown that $\bar{x} = f'(x')$.

CHAPTER II

FIBRATIONS AND COFIBRATIONS

1. The Fundamental Group

In this section the fundamental group of a space will be constructed. This group will allow us to transform certain topological problems into algebraic ones.

Definition 1. Let X be a space. A loop in X is a path ω: $I \to X$ such that $\omega(0) = \omega(1)$; i.e. a loop is a closed path.

Throughout the remainder of this section all paths considered will be loops in X with end and origin at x_0.

As before, if ω_1 and ω_2 are homotopic, then we shall write $\omega_1 \simeq \omega_2$. This is easily seen to define an equivalence relation on the loops in X, in fact, on all paths in X. The equivalence class containing ω will be denoted by $[\omega]$.

Given loops ω_1 and ω_2 in X, define $[\omega_1] * [\omega_2]$ to be the equivalence class $[\omega_1 * \omega_2]$. We then have

Proposition 1. Given loops ω_1 and ω_2 in X, the operation defined by $[\omega_1] * [\omega_2] = [\omega_1 * \omega_2]$ is well defined.

Proof. We must show that if $\bar{\omega}_1 \simeq \omega_1$ and $\bar{\omega}_2 \simeq \omega_2$ then $\bar{\omega}_1 * \bar{\omega}_2 \simeq \omega_1 * \omega_2$.

So let F_1: $\bar{\omega}_1 \simeq \bar{\omega}_2$ and F_2: $\omega_1 \simeq \omega_2$.

Define

$$G(t,t') = \begin{cases} F_1(2t,t') & \text{if } 0\le t\le 1/2 \\ F_2(2t-1,t') & \text{if } 1/2\le t\le 1 \end{cases}$$

Since $F_1(1,t') = \tilde{x}_0 = F_2(0,t')$ for all $t' \varepsilon I$, it follows that G is continuous.

Now,

$$G(t,0) = \begin{cases} F_1(2t,0) \\ F_2(2t-1,0) \end{cases} = \begin{cases} \bar{\omega}_1(2t) \\ \bar{\omega}_2(2t-1) \end{cases} = \bar{\omega}_1 * \bar{\omega}_2(t)$$

and

$$G(t,1) = \begin{cases} F_1(2t,1) \\ F_2(2t-1,1) \end{cases} = \begin{cases} \omega_1(2t) \\ \omega_2(2t-1) \end{cases} = \omega_1 * \omega_2(t)$$

Also $G(0,t') = F_1(0,t') = x_0$ and $G(1,t') = F_2(1,t') = x_0$ for all $t' \varepsilon I$.

Therefore $G\colon \bar{\omega}_1 * \bar{\omega}_2 \simeq \omega_1 * \omega_2$.

Note: This result is true for any paths ω_1 and ω_2 such that $\omega_1(1) = \omega_2(0)$.

Proposition 2. Let ω_1, ω_2 and ω_3 be paths (not necessarily closed) in X such that $\omega_1(1) = \omega_2(0)$ and $\omega_2(1) = \omega_3(0)$. Then $(\omega_1 * \omega_2) \times \omega_3 \simeq \omega_1 * (\omega_2 * \omega_3)$.

Proof. Define a homotopy $G\colon I \times I \to X$ by

$$G(t,t') = \begin{cases} \omega_1 \dfrac{4t}{t'+1} & \text{if } 0 \le t \le \dfrac{t'+1}{4} \\ \omega_2(4t-t'-1) & \text{if } \dfrac{t'+1}{4} \le t \le \dfrac{t'+2}{4} \\ \omega_3\left(\dfrac{4t-2-t'}{2-t'}\right) & \text{if } \dfrac{t'+2}{4} \le t \le 1. \end{cases}$$

Then,

$$G(t,0) = \begin{cases} \omega_1(4t) & \text{if } 0 \le t \le \frac{1}{4} \\ \omega_2(4t-1) & \text{if } \frac{1}{4} \le t \le \frac{1}{2} \\ \omega_3(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases} = \big((\omega_1 * \omega_2) * \omega_3\big)(t)$$

$$G(t,1) = \begin{cases} \omega_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \omega_2(4t-2) & \text{if } \frac{1}{2} \le t \le \frac{3}{4} \\ \omega_3(4t-3) & \text{if } \frac{3}{4} \le t \le 1 \end{cases} = \big(\omega_1 * (\omega_2 * \omega_3)\big)(t)$$

Also, $G(0,t') = \omega_1(0)$ and $G(1,t') = \omega_3\left(\dfrac{2-t'}{2-t'}\right) = \omega_3(1)$ for all $t' \in I$. Thus we shall be finished if G is continuous.

$$\omega_1\left(\frac{4\left(\frac{t'+1}{4}\right)}{t'+1}\right) = \omega_1(1) = \omega_2(0) = \omega_2\left(4\left(\frac{t'+1}{4}\right) - t' - 1\right)$$

for all $t' \in I$.

Also,

$$\omega_2\left(4\frac{(t'+2)}{4} - t' - 1\right) = \omega_2(1) = \omega_3(0) = \omega_3\left(\frac{4\left(\frac{t'+2}{4}\right) - 2 - t'}{2 - t'}\right)$$

for all $t' \in I$. It then follows that G is continuous.

Corollary 2.1 The operation * in Proposition 1 is associative on the set of equivalence classes.

Proposition 3 If C_x: $I \to X$ denotes the constant path at x, then $[\omega * C_{x_o}] = [\omega] = [C_{x_o} * \omega]$ for any loop ω in X.

Proof. Define F: $I \times I \to X$ by

$$F(t,t') = \begin{cases} \omega\left(\frac{2t}{t'+1}\right) & \text{if } 0 \le t \le \frac{t'+1}{2} \\ x_o & \text{if } \frac{t'+1}{2} \le t \le 1 \end{cases}$$

Since $\omega\left(\frac{2\frac{(t'+1)}{2}}{t'+1}\right) = \omega(1) = x_o$ for all t', it follows that F is continuous.

$$F(t,o) = \begin{bmatrix} \omega(2t) & \text{if } 0 \le t \le 1/2 \\ x_o & \text{if } 1/2 \le t \le 1 \end{bmatrix} = \omega * C_{x_o}(t)$$

$F(t,1) = \omega(t)$. Also, $F(0,t') = \omega(0) = x_o = F(1,t')$ for all $t' \varepsilon I$. Thus $F = \omega * C_{x_o} \simeq \omega$. We can show in a similar manner that $C_{x_o} * \omega \simeq \omega$.

Proposition 4 Let P(X) be the set of equivalence classes of loops at x_o in X. Then $(P(X),*)$ is a group.

Proof. By Propositions 1, 2, and 3, * is an associative operation and $[C_{x_o}]$ is the identity map. We need only prove that $[\omega]$ has an inverse for all $[\omega] \varepsilon P(X)$.

If $[\omega] \varepsilon P(X)$, define $[\omega]^{-1} = [\omega^{-1}]$ (recall that $\omega^{-1}(t) = \omega(1-t)$).

Define $F: I \times I \to X$ by

$$F(t,t') = \begin{cases} \omega((1-t')2t) & \text{if } 0 \leq t \leq 1/2 \\ \omega((2-2t)(1-t')) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Since

$$\omega\big((1-t')2(1/2)\big) = \omega(1-t')$$

$$= \omega\big((2-2(1/2)\big)(1-t')$$

for all $t' \varepsilon I$, it follows that F is continuous.

$$F(t,0) = \begin{cases} \omega(2t) & \text{if } 0 \leq t \leq 1/2 \\ \\ \omega(2-2t) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

$$= \begin{cases} \omega(2t) \\ \\ \omega^{-1}(2t-1) \end{cases} = \omega*\omega^{-1}(t)$$

$F(t,1) = \omega(0) = x_o$. Also $F(0,t') = \omega(0) = x_o$ and $F(0,t') = \omega(0) = x_o$. Thus $F\colon \omega*\omega^{-1} \simeq C_{x_o}$ which proves that $[\omega^{-1}]$ is a right inverse of $[\omega]$. This is enough to show that $[\omega^{-1}]$ is the inverse of $[\omega]$.

This group is the first homotopy group or fundamental group of the space X at the point x_o and is denoted by $\pi(X,x_o)$.

Since any closed path at x_o (and any homotopy between closed path) must lie in the path component A of X containing x_o, it follows that $\pi(A,x_o)$ is isomorphic to $\pi(X,x_o)$. Hence the fundamental group can give information about path components alone. It is easy to show that if X is path connected then $\pi(X,x_o)$ is isomorphic to $\pi(X,x_1)$ for any $x_o,x_1 \in X$ (see [3], p. 151).

2. The Homotopy Extension Property

In Section 1 of Chapter I a generalization of the extension problem was formulated. The new problem was then called the generalized extension problem. By relating the concept of homotopy to the extension problem a new problem may be obtained. The <u>homotopy extension problem</u> can be stated as follows:

Given a map $f\colon A \to Y$ where $A \subset X$, does there exist a map $h\colon X \to Y$ such that h/A is homotopic to f. We also have the generalized homotopy extension problem: given maps $f\colon X \to Y$ and $g\colon X \to Z$, does there exist $h\colon Y \to Z$ such that hf is homotopic to g.

We begin by extending the concept of homotopy to arbitrary mappings.

Definition 1. Let $f,g\colon X \to Y$ be mappings. Then f is said to be homotopic to g relative to $A \subset X$, and we write $f \simeq g$ rel A, if there exists a map $F\colon X \times I \to Y$ such that $F(x,o) = f(x)$ and $F(x,1) = g(x)$ for $x \in X$; and such that $F(a,t) = f(a)$ for all $(a,t) \in A \times I$.

Remark. If ω_1 and ω_2 are paths in X, we wrote in the previous chapter $\omega_1 \simeq \omega_2$ to denote the fact $\omega_1 \simeq \omega_2$ rel $\{0,1\}$. We shall continue to use this notation for paths alone.

Definition 2. Let X be a space and let $A \subset X$. The pair (X,A) is said to have the homotopy extension property with respect to a space Y if, given maps $g\colon X \to Y$ and $G\colon A \times I \to Y$ such that $g(x) = G(x,o)$ for $x \in A$, there is a map $F\colon X \times I \to Y$ such that $F(x,o) = g(x)$ for $x \in X$ and $F/A \times I = G$.

Definition 3. A map $f\colon X \to Y$ is called a cofibration if, given maps $g\colon Y \to Z$ and $G\colon X \times I \to Z$ (where Z is arbitrary) such that $gf(x) = G(x,o)$ for $x \in X$, there is a map $F\colon Y \times I \to Z$ such that $F(y,o) = g(y)$ for $y \in Y$ and $F(f(x),t) = G(x,t)$ for $x \in X$, $t \in I$. If f is regarded as a map from $X \times o$ to $Y \times o$ and g is regarded as a map from $Y \times o$ to Z, the existence of F is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram everywhere commutative.

$$\begin{tikzcd}
X\times o \arrow[rr, "c"] \arrow[dd, "f"'] & & X\times I \arrow[dl, "G"'] \arrow[dd, "f\times 1_I"] \\
& Z & \\
Y\times o \arrow[ur, "g"] \arrow[rr, "c"] & & Y\times I \arrow[ul, dashed, "F"']
\end{tikzcd}$$

Remark. It is easy to see that the inclusion map i: $A \subset X$ is a cofibration if and only if (X,A) has the homotopy extension property with respect to any space Y.

Proposition 1. If f: $X \to Y$ is a cofibration and α: $X \to Z$ is a map, the existence of g: $Y \to Z$ such that $gf \simeq \alpha$ implies the existence of g': $Y \to Z$ such that $g' \simeq g$ and $g'f = \alpha$.

Proof. Let G: $X\times I \to Z$ be a homotopy from gf to α. Then $G(x,o) = gf(x)$.

Since f is a cofibration, there exists a map F: $Y\times I \to Z$ such that $F(y,o) = g(y)$ for $y \in Y$ and $F(f(x),t) = G(x,t)$, for $(x,t) \in X\times I$.

Now, $G(x,1) = \alpha$. Therefore $F(f(x),1) = G(x,1) = \alpha(x)$.

Define g': $Y \to Z$ by $g(y) = F(y,1)$ for $y \in Y$. Then $g'f = \alpha$ and F is a homotopy from g to g'.

It will be shown next that this generalized homotopy extension problem is equivalent to a homotopy extension problem. To do this, we first construct a space called the mapping cylinder.

Let g: $X \to Z$ be a map and let $X\times I \vee Z = W$ denote the disjoint union of $X\times I$ and Z. Define an equivalence relation on W as follows: $(x,1) \simeq g(x)$ for $x \in X$, and $(x,1) \sim (x',1)$ if and only if $g(x) = g(x')$.

Let Z_g be the space of all equivalence classes in W with the identification topology induced by the map p: $W \to Z_g$ which sends each element in W to its equivalence class in Z_g.

We use $[x,t]$ to denote the point of Z_g corresponding to $(x,t) \in X\times I$ under p and $[z]$ to denote the point of Z_g corresponding to $z \in Z$ (thus $[x,1] = [g(x)]$).

Define i: $X\to Z_g$ and j: $Z\to Z_g$ by $i(x) = [x,o]$ and $j(z) = [z]$.

Proposition 2. The mappings i and j are imbeddings.

Proof. If i': $X\to X\times I\vee Z$ and j': $Z\to X\times I\vee Z$ are the natural imbeddings, we see that $i = pi'$ and $j = pj'$. Therefore i and j are continuous and one to one maps.

If U is a closed set in Z then $p^{-1}j(U) = U\cup(g^{-1}(U)\times 1)$ which is closed in $X\times I\vee Z$ because g is continuous. Thus j(U) is closed in Z_g, and hence j is a closed map.

If V is a closed set in X then $i(V) = \{[x,o] \mid (x,o) \in V\times o\}$. This set is obviously closed in Z_g. Therefore i is also a closed map.

Proposition 3. Let g: $X\to Z$ be a map and Z_g be the mapping cylinder of g. Then the function r: $Z_g\to Z$ defined by $r([x,t]) = g(x)$ for $x \in X$ and $r([z]) = z$ for $z \in Z$ is a retract.

Proof. It remains to prove that r is continuous.

Define h: $W\to Z$ by $h(x,t) = g(x)$ if $(x,t) \in X\times I$ and $h(z) = z$ if $z \in Z$. Clearly $h/X\times I$ and h/Z are both continuous maps. Also $X\times I$ and Z are closed in W, $X\times I\vee Z = W$ and $X\times I\cap Z = \phi$. Therefore h is itself continuous.

But $rp = h$. Since p is an identification, it follows from Lemma 1 in Chapter I, Section 1 that r is continuous.

Proposition 4. For any given map $g\colon X \to Y$ the maps $i\colon X \to Z_g$ and $pg\colon X \to Z_g$ are homotopic.

Proof. Define $F\colon X \times I \to Z_g$ by

$$F(x,t) = p(x,t) \quad \text{for } (x,t) \in X \times I.$$

Clearly, F is continuous. Also $F(x,o) = p(x,o) = i(x)$ and $F(x,1) = p(x,1) = [g(x)] = pg(x)$. We now prove the main result in this section.

Proposition 5. Given any two maps $g\colon X \to Z$ and $f\colon X \to Y$, the following two statements are equivalent:

(i) There is a map $h\colon Y \to Z$ such that $hf \simeq g$.

(ii) There is a map $F\colon Z_f \to Z$ such that $Fi \simeq g$.

Proof (i) $\Rightarrow$ (ii). According to Proposition 3 there is a retract $r\colon Z_f \to Y$ given by $r([x,t]) = g(x)$ if $(x,t) \in X \times I$ and $r([y]) = y$ if $y \in Y$. Let $F = hr\colon Z_f \to Z$. By Proposition 4 we have $Fi = hri \simeq hrpf = hf \simeq g$.

(ii) $\Rightarrow$ (i). Let $h = F/Y$. Then, by Proposition 4 we have $hf = Fpf \simeq Fi \simeq g$.

Corollary 5.1 If $f\colon X \to Y$ is a cofibration and $g\colon X \to Z$ is a map, the following two statements are equivalent.

(i) There is a map $h'\colon Y \to Z$ such that $h'f = g$.

(ii) There is a map $F\colon Z_f \to Z$ such that $Fi \simeq g$.

Proof (i) $\Rightarrow$ (ii). Same as in Proposition 5.

(ii) $\Rightarrow$ (i). By Proposition 5 we know that $hf \simeq g$ where $h = F/Y$.

By Proposition 1 there exists $h'\colon Y \to Z$ such that $h' \simeq h$ and $h'f = g$.

Remark. It can be shown (see [3] 14.12) that the imbedding $i\colon X \to Z_f$ is a cofibration, so that (ii) above can be restated to read as follows:

(ii) There is a map $F\colon Z_f \to Z$ such that $Fi = g$.

3. The Homotopy Lifting Property

In order that the lifting problem of a map be a problem in the homotopy class of the map we need an analogue of the homotopy extension property, called the homotopy lifting property defined as follows.

Definition 1. A map $\tilde{p}\colon X \to Y$ is said to have the homotopy lifting property with respect to a space X if, given maps $f'\colon X \to \tilde{X}$ and $F\colon X \times I \to Y$ such that $F(x,o) = pf'(x)$ for all $x \in X$, there is a map $G\colon X \times I \to \tilde{X}$ such that $G(x,o) = f'(x)$ for $x \in X$ and $F = pG$.

Thus, if f' is regarded as a map from $X \times o$ to $\tilde{X}$, the existence of G is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram commutative (where i is the inclusion map)

$$\begin{tikzcd} X \times o \arrow[r, "f'"] \arrow[d, "i"'] & \tilde{X} \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dashed, "G"] & Y \end{tikzcd}$$

Definition 2. A map p: $\tilde{X} \to Y$ is said to be a <u>fibration</u> if p has the homotopy lifting property with respect to any space X.

It is easy to verify that if p is a fibration and $f \simeq g$, where f and g are maps from some space X into Y, then f can be lifted to $\tilde{X}$ if and only if g can be lifted to $\tilde{X}$. Thus if p is a fibration, whether or not f can be lifted to $\tilde{X}$ is a property of the homotopy class of the map.

Proposition 1. Let p: $\tilde{X} \to Y$ be a fibration and let $\omega: I \to Y$ be a path in Y such that $\omega(o) \in p(X)$. Then ω can be lifted to $\tilde{X}$.

Proof. Let $\tilde{x}_o \in \tilde{X}$ be such that $p(\tilde{x}_o) = \omega(o)$. If we let $\{\tilde{x}_o\}$ be the one point set, we have the following diagram:

$$\begin{tikzcd}
\{\tilde{x}_o\} \times o \arrow[r, "C"] \arrow[d, "i"'] & \tilde{X} \arrow[d, "p"] \\
\{x_o\} \times I \arrow[r, "\omega"] \arrow[ur, dashed, "\tilde{\omega}"] & Y
\end{tikzcd}$$

The existence of $\tilde{\omega}$ follows from the fact that p is a fibration.

Clearly, $\tilde{\omega}$ is the desired lifting.

The proposition above suggests the following result.

Proposition 2. A map p: $\tilde{X} \to Y$ is a fibration if and only if there exists a lifting function $\lambda: \bar{Y} \to \tilde{X}^I$ for p.

Proof. The proof involves repeated use of Theorem 2 in Chapter 0.

Suppose p is a fibration. Define f': $\bar{Y}\to\tilde{X}$ and F: $\bar{Y}\times I\to Y$ by

$$f'(\tilde{x},\omega) = \tilde{x} \quad \text{and} \quad F\big((\tilde{x},\omega),t\big) = \omega(t)$$

Then f' is clearly continuous. Also, F is given by the composition

$$\bar{Y} \times I \xrightarrow{p_2\times 1_I} Y^I \times I \xrightarrow{E} Y$$

where $p_2 = \bar{Y}\to Y^I$ is the projection onto the second factor. Since p_2 is continuous it follows from Theorem 2 in Chapter 0 that F is also continuous.

Then $F\big((\tilde{x},\omega),o\big) = \omega(o) = p(\tilde{x}) = p\circ f'(\tilde{x},\omega)$. By the homotopy lifting property of p, there is a map F': $\bar{Y}\times I\to\tilde{X}$ such that $p\circ F' = F$ and $F'\big((\tilde{x},\omega),0\big) = f'(\tilde{x},\omega)$. Define λ: $\bar{Y}\to\tilde{X}^I$ by

$$\lambda(\tilde{x},\omega)(t) = F'\big((\tilde{x},\omega),t\big).$$

Then, F' is given by the composition

$$\bar{Y} \times I \xrightarrow{\lambda\times 1_I} \tilde{X}^I \times I \xrightarrow{E} X.$$

By Theorem 2 in Chapter 0, it follows that λ is continuous. It is clear that λ is the desired lifting function.

Suppose now that p has a lifting function λ: $\bar{Y} \to \tilde{X}^I$.

Let f': $X \to \tilde{X}$ and F: $X \times I \to Y$ be maps such that $pf'(x) = F(x,o)$. Define g: $X \to X^I$ by $(g(x))(t) = F(x,t)$. Since F is the composition $X \times I \xrightarrow{g \times 1_I} Y^I \times I \xrightarrow{E} Y$, it follows from Theorem 2 of the introduction that g is continuous.

But $pf'(x) = F(x,o) = (g(x))(o)$. Therefore $(f'(x),g(x)) \in \bar{Y}$.

Define F': $X \times I \to \tilde{X}$ as the composition

$$X \times I \xrightarrow{(f',g) \times 1_I} \bar{Y} \times I \xrightarrow{\lambda \times 1_I} \tilde{X}^I \times I \xrightarrow{E} \tilde{X}.$$

Again, by Theorem 2 of the introduction, F' is continuous.

But

$$F'(x,o) = \lambda(f'(x),g(x))(o)$$

$$= f'(x)$$

and

$$pF'(x,t) = p\big(\lambda(f'(x),g(x))\big)(t) = (g(x))(t)$$

$$= F(x,t).$$

Thus p is a fibration.

Example. We showed in Proposition 6, Section 2, Chapter I, that a covering projection has a lifting function. From the proposition above it follows that a covering projection is a fibration.

Let ω_1 and ω_2 be homotopic paths in $\tilde{X}$ and $p\colon \tilde{X} \to Y$ be a map. Let $F\colon I \times I \to \tilde{X}$ be a homotopy from ω_1 to ω_2. Then pF is clearly a homotopy from $p\omega_1$ to $p\omega_2$. We then have

Definition 3. Let $p\colon (\tilde{X}, x_o) \to (Y, y_o)$ be a map. Define $p_\#\colon \pi(\tilde{X}, \tilde{x}_o) \to \pi(Y, y_o)$ by $p_\#([\omega]) = [p\omega]$. From the observation above it follows that $p_\#$ is well defined. It is also easy to show that $p_\#$ is a homomorphism.

Proposition 3. If $p\colon (\tilde{X}, \tilde{x}_o) \to (Y, y_o)$ is a fibration with unique path lifting, $p_\#$ is a monomorphism.

Proof. By Proposition 2 it follows that p has a lifting function. Then, by Lemma 1, Section 2, in Chapter I, we have that if $p\tilde{\omega}_1 \simeq p\tilde{\omega}_2$, $\lambda(\tilde{x}_o, p\tilde{\omega}_1)$ is homotopic to $\lambda(\tilde{x}_o, p\tilde{\omega}_2)$. Since p has the unique path lifting property, it follows that $\lambda(\tilde{x}_o, p\tilde{\omega}_1) = \tilde{\omega}_1$ and $\lambda(\tilde{x}_o, p\tilde{\omega}_2) = \tilde{\omega}_2$. Therefore $\tilde{\omega}_1 \simeq \tilde{\omega}_2$ and whence the proposition.

This last result substantially increases the usefulness of the next proposition known as the lifting theorem.

Proposition 4. Let $p\colon (\tilde{X}, x_o) \to (Y, y_o)$ be a fibration with unique path lifting and let $f\colon (\bar{X}, x_o) \to (Y, y_o)$ be a map. Assume that X is path connected and locally path connected. A necessary and sufficient condition that f have a lifting to $(\tilde{X}, \tilde{x}_o)$ is that in $\pi(Y, y_o)$

$$f_\#(\pi(X, x_o)) \subset p_\#(\pi(\tilde{X}, \tilde{x}_o))$$

Proof. The condition is clearly necessary for, if a lifting f' for f exists, then $pf' = f$. Therefore

$$\begin{aligned} f_{\#}(\pi(X,x_o)) &= (pf')_{\#}(\pi(X,x_o)) \\ &= p_{\#}f'_{\#}(\pi(X,x_o)) \\ &\subset p_{\#}(\pi(\tilde{X},\tilde{x}_o)). \end{aligned}$$

We show that the converse is also true.

Since p is a fibration, there is a lifting function λ. Also p has the unique path lifting property. Furthermore, because of the condition, given any closed path ω at x_o in X, there exists a closed path $\tilde{\omega}$ at $\tilde{x}_o$ in $\tilde{X}$ such that $f\omega \simeq p\tilde{\omega}$. The result now follows from Chapter I, Section 2, Proposition 7.

Examples. If S^1 is the unit circle, it may be shown that $\pi(S^1,p_o)$ is isomorphic to the additive group of integers Z (see [1], I-8). The exponential function exp: $R^1 \to S^1$ is easily shown to be a covering projection. Since R^1 is contractible, $\pi(R^1,o) = \{o\} \quad Z$. Thus if a map $f\colon X \to S^1$ has a lifting $f'\colon X \to R^1$ and X is path connected and locally path connected, then f maps every path in X to a path homotopic to the constant path.

Example. Let $p\colon (\tilde{X},\tilde{x}_o) \to (S^1,1)$ be a fibration and let $p_n\colon (S^1,1) \to (S^1,1)$ be the map given by $p_n(z) = z^n$. It can be shown that $p_{n_{\#}}(\pi(S^1,1))$ is the subgroup of Z generated by n. Since non-trivial subgroups of Z are infinite cyclic, Proposition 3 implies that a necessary condition for p_n to have a lifting to $(\tilde{X},\tilde{x}_o)$ is that $\pi(\tilde{X},\tilde{x}_o) \approx Z$. By the lifting theorem it follows that for all multiples mn of some integer n, p_{mn} can be lifted to $(\tilde{X},\tilde{x}_o)$.

Corollary 4.1 Let p: $(\tilde{X},\tilde{x}_o) \to (Y,y_o)$ be a fibration with unique path lifting. If Y is path connected and locally path connected, then there exists a section of p if and only if $p_{\#}(\pi(\tilde{X},\tilde{x}_o)) = \pi(Y,y_o)$.

Proof. If $p_{\#}(\pi(\tilde{X},\tilde{x}_o)) = \pi(Y,y_o)$ it follows from Proposition 4 that 1_y: $(Y,y_o) \to (Y,y_o)$ has a lifting f' to $(\tilde{X},\tilde{x}_o)$. Then $pf' = 1_y$; i.e., f' is a section of p.

If p has a section f': $(Y,y_o) \to (\tilde{X},\tilde{x}_o)$ then $1_Y = pf'$. Therefore

$$\begin{aligned}\pi(Y,y_o) &= 1_{Y_{\#}}(\pi(Y,y_o)) \\ &= p_{\#}f'_{\#}(\pi(Y,y_o)) \\ &\subset p_{\#}(\pi(\tilde{X},\tilde{x}_o))\end{aligned}$$

$$\therefore\ \pi(Y,y_o) = p_{\#}(\pi(\tilde{X},\tilde{x}_o)).$$

CHAPTER III

LOCAL FIBRATIONS

Let $p\colon E \to B$ be a map and let W be a subset of B^I. Define $\tilde{W} \subset E \times W \times I$ to be the set

$$\tilde{W} = \{(e,\omega,s) \mid \omega(s) = p(e)\}.$$

An extended lifting function over W is a map $\Lambda\colon \tilde{W} \to E^I$ such that $p\Lambda(e,\omega,s)(t) = \omega(t)$ and $\Lambda(e,\omega,s)(s) = e$.

Lemma 1. A map $p\colon E \to B$ has a lifting function if and only if it has an extended lifting function over B^I.

Proof. Suppose p has an extended lifting function over B^I. Then, a lifting function λ for p is given by $\lambda(e,\omega) = \Lambda(e,\omega,o)$ for $(e,\omega) \in \bar{B}$.

To prove the converse, given a path ω in B, let ω_s and ω^s be the paths defined by

$$\omega_s(t) = \begin{cases} \omega(s-t) & \text{if } 0 \le t \le s \\ \omega(o) & \text{if } s \le t \le 1 \end{cases}$$

and

$$\omega^s(t) = \begin{cases} \omega(s+t) & \text{if } 0 \le t \le 1 - s \\ \\ \omega(1) & \text{if } 1 - s \le t \le 1 \end{cases}$$

Then $\omega_s(o) = \omega(s) = \omega^s(o)$. Next define $f_o\colon B^I \times I \to B^I$ and $f^o\colon B^I \times I \to B^I$ by $f_o(\omega,s) = \omega_s$ and $f^o(\omega,s) = \omega^s$. We show that f_o and f^o are continuous.

Define $J_1 \subset I \times I$ to be the set $J_1 = \{(s,t)|t \le s\}$ and let $J_2 \subset I \times I$ be the set $J_2 = \{(s,t)|t \ge s\}$. Clearly $J_1 \cup J_2 = I \times I$. Define $k_1\colon J_1 \to I$ by $k(s,t) = s-t$ and $k_2\colon J_2 \to I$ by $k_2(s,t) = 0$. Clearly, k_1 and k_2 are continuous.

Let $B_1 = \{(\omega,s,t)|t \le s, \omega \varepsilon B^I\}$ and $B_2 = \{(\omega,s,t)|s \le t, \omega \varepsilon B^I\}$. Then $B_1 \quad B_2 = B^I \times I \times I$ and B_1 and B_2 are closed subsets of $B^I \times I \times I$.

Define $F_1 = B_1 \to B$ and $F_2\colon B_2 \to B$ as the compositions $B^I \times J_1 \xrightarrow{1_{B^I} \times k_1} B^I \times I \xrightarrow{E} B$ and $B^I \times J_2 \xrightarrow{1_{B^I} \times k_2} B^I \times I \xrightarrow{E} B$, respectively. Then $F_1(\omega,s,t) = \omega(s-t)$ and $F_2(\omega,s,t) = \omega(o)$ are both continuous functions on B_1 and B_2, respectively. Since $F_1/B_1 \cap B_2 = F_2/B_1 \cap B_2$, we may define a continuous function $F\colon B^I \times I \times I \to B$ such that $F/B_1 = F_1$ and $F/B_2 = F_2$.

But the composition $(B^I \times I) \times I \xrightarrow{f_o \times 1_I} B^I \times I \xrightarrow{E} B$ is exactly F. From Theorem 2 in Chapter 0 it follows that f_o is continuous. In a similar manner we may show that f^o is also continuous.

Let λ be the lifting function for p, and define $\Lambda\colon B^I \to E^I$ by

$$\Lambda(e,\omega,s)(t) = \begin{cases} \lambda(e,\omega_s)(s-t) & \text{if } 0 \le t \le s \\ \\ \lambda(e,\omega^s)(t-s) & \text{if } s \le t \le 1 \end{cases}$$

Then $\Lambda(e,\omega,s)(s) = e$ and $p\Lambda(e,\omega,s)(t) = \omega(t)$.

We shall prove that Λ is continuous, thereby establishing Λ as an extended lifting function.

Define $J_1 \subset I\times I$ and $J_2 \subset I\times I$ as before. Let $D_1 = E \times B^I \times J_1 \cap \widetilde{B^I} \times I$. Clearly D_1 is closed in $\widetilde{B^I} \times I \subset E \times B^I \times I \times I$.

Define $k_1 = J_1 \to I$ by $k_1(s,t) = s-t$ and let p_1: $I\times I \to I$ by the first projection. Then the function (p_1,k_1): $J_1 \to I\times I$ defined as

$$(p_1,k_1)(s,t) = (p_1(s,t),k_1(s,t))$$

$$= (s,s-t)$$

is continuous.

Define F_1: $D_1 \to E$ as the composition

$$D_1 \xrightarrow{I_E \times 1_{B^I} \times (p_1,k_1)} E\times B^I \times I \times I \xrightarrow{1_E \times f_o \times 1_I} \tilde{B}\times I \xrightarrow{\lambda \times 1_I} E^I \times I \xrightarrow{E} E$$

Every one of these functions is continuous and hence F_1 is continuous.

We may show in exactly the same manner that the function F_2: $D_2 \to E$ defined by $F_2(e,\omega,s,t) = \lambda(e,\omega^s)(t-s)$ is also continuous where $D_2 = E \times B^I \times J_2 \subset \widetilde{B^I} \times I$ and $J_2 = \{(s,t) | t \ge s\}$.

Clearly $D_1 \bigcup D_2 = B^I \times I$ and D_1 and D_2 are closed sets. Also, if $s = t$ then

$$F_1(e,\omega,s,t) = \lambda(e,\omega_s)(0) = e = \lambda(e,\omega^s)(0) = F_2(e,\omega,s,t).$$

We may then define a continuous F on $B^I \times I$ by setting $F/D_1 = F_1$ and $F/D_2 = F_2$.

But the composition $B^I \times I \xrightarrow{\Lambda\times 1_I} E^I \times I \xrightarrow{E} E$ is exactly F. Therefore, by Theorem 2 in Chapter 0, it follows that Λ is continuous.

We now make use of the theorems about cozero sets stated in Chapter 0 to prove the following proposition from [2], 3.2, Chapter XX.

Proposition 1. Let $p\colon E \to B$ be a map and assume that there is a nbd-finite covering $\{U_\beta | \beta\varepsilon \mathcal{B}\}$ of B by cozero sets such that each $p/p^{-1}(U_\beta)\colon p^{-1}(U_\beta) \to U_\beta$ is a fibration. Then p is a fibration.

Proof. We shall construct a lifting function for p.

For each finite set $\{\beta_1,\ldots,\beta_n\}$ of indices, define $W(\beta_1,\ldots,\beta_n) = \{\alpha | \alpha\varepsilon B^I$ and $\alpha(t) \,\varepsilon\, U_{\beta_k}$ for $t \,\varepsilon\, I_k = \left[\frac{k-1}{n}, \frac{k}{n}\right]\}$. Then $W(\beta_1,\ldots,\beta_n) = \bigcap_{i=1}^{n} (I_i, U_{\beta_i})$. Since U_{β_i} is a cozero set, it follows from Theorem 4(c) in Chapter 0 that (I_i, U_{β_i}) is also a cozero set. By Theorem 4(a) in Chapter 0, $W(\beta_1,\ldots,\beta_n)$ is itself a cozero set.

For each positive integer n let

$$\mathcal{W}_n = \{W(\beta_1,\ldots,\beta_n) | (\beta_1,\ldots,\beta_n) \;\varepsilon\; \mathcal{B}\times\mathcal{B}\times \cdots \times\mathcal{B}(\text{n-times})\}$$

Then $\bigcup_{n\geq 1} \mathcal{W}_n = \mathcal{W}$ is clearly an open covering of B^I. Furthermore, each family $\mathcal{W}_n$ is nbd-finite: given $\lambda \in B^I$, the compact $\alpha(I) \subset B$ has a nbd V meeting at most finitely many U_β, so the nbd (I,V) of α can meet at most finitely many sets in $\mathcal{W}_n$. According to Theorem 5 in Chapter 0, $\mathcal{W}$ has a nbd-finite refinement $\{W_\mu | \mu \in \mathcal{M}\}$ by cozero sets.

For each W_μ let $D_\mu \subset E \times W_\mu \times I$ be the subspace $D_\mu = \{(e,\alpha,s) | p(e) = \alpha(s)\}$. We next construct a continuous $\lambda_\mu : D_\mu \to E^I$ such that $\lambda_\mu(e,\alpha,s)(s) = e$ and $p\lambda_\mu(e,\alpha,s)(t) = \alpha(t)$ for each $(e,\alpha,s) \in D_\mu$, $t \in I$.

Given W_μ select some $W(\beta_1,\ldots,\beta_n) \subset W_\mu$ and for each $i=1,2,\ldots,n$ let Λ_i be an extended lifting function for the fibration $p/p^{-1}(U_{\beta_i})$: $p^{-1}(U_{\beta_i}) \to U_{\beta_i}$ which exists by Lemma 1.

For each $\alpha \in W_\mu$ let α_i be the path that agrees with α on I_i and is constant elsewhere. Let $(e,\alpha,s) \in D_\mu$ and let k be an integer such that $s \in I_k$. Then, because $p(e) = \alpha(s)$ so that $e \in p^{-1}(U_{\beta_k})$, we set

$$\omega(t) = \begin{cases} \Lambda_k(e,\alpha_k,s)(t) & \text{if } t \in I_k \\ \Lambda_{k-1}\left(\omega\left(\frac{k-1}{n}\right),\alpha_{k-1}, \frac{k-1}{n}\right)(t) & \text{if } t \in I_{k-1} \\ \Lambda_{k+1}\left(\omega\left(\frac{k}{n}\right),\alpha_{k+1}, \frac{k}{n}\right)(t) & \text{if } t \in I_{k+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \end{cases}$$

and radiate outward. This path is uniquely determined. Also if $s = \frac{k}{n}$, so that $s \in I_k \cap I_{k+1}$, then it is easy to see that the same path ω is obtained, when Λ_{k+1} is taken as the starting point; similarly for the case $s = \frac{k-1}{n} \in I_k \cap I_{k-1}$.

We next show that $\lambda_\mu : D_\mu \to E^I$ defined by $\lambda_\mu(e,\alpha,s)(t) = \omega(t)$ is continuous. For each $i < k$, $\Lambda\left(\omega\left(\frac{i}{n}\right), \alpha_i, \frac{i}{n}\right)\Big| I_i$ is continuous. Also, if $i > k$, $\Lambda_i\left(\omega\left(\frac{i-1}{n}\right), \alpha_i, \frac{i-1}{n}\right)\Big| I_i$ is continuous. From the way in which they are defined, it is clear that these n pieces fit together. Using the same method as in Lemma 1, after a lengthy computation we see that the composition $D_\mu \times I \xrightarrow{\lambda_\mu \times 1_I} E^I \times I \xrightarrow{E} E$, which sends (e,ω,s,t) into $\omega(t)$, is continuous. It follows from Theorem 2 in Chapter 0 that λ_μ is continuous and it is easy to see that λ_μ satisfies the other requirements of a generalized lifting function.

Select a definite open covering $\{U\}$ of B^I such that each U meets only finitely many sets W_μ. For each U let $\bar{B}_U = \bar{B} \cap (E \times U) \subset E \times B^I$. Well order the indexing set M and choose a definite characterizing function $C_\mu : W_\mu \to I$ for the cozero set, W_μ. Given $U \in \{U\}$, let $\mu_1 < \mu_2 < \cdots < \mu_n$ be the indices of all sets W_μ meeting U, and define n continuous real-valued functions on U by

$$t_r(\alpha) = \sum_{i=1}^{r} C_{\mu_i}(\alpha) / \sum_{i=1}^{n} C_{\mu_i}(\alpha) \qquad r=1,\ldots,n.$$

Define $\lambda_U : \bar{B}_U \to E^I$ by:

$$\lambda_U(e,\alpha)(t) = \lambda_{\mu_1}(e,\alpha,o)(t) \qquad \text{if } o\leq t\leq t_1(\alpha)$$

$$= \lambda_{\mu_2}(\lambda_U(e,\alpha)(t_1(\alpha)),\alpha,t_1(\alpha)) \quad \text{if } t_1(\alpha)\leq t\leq t_2(\alpha)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Then $\lambda_U(e,\alpha)(o) = e$ and $p\lambda_U(e,\alpha)(t) = \alpha(t)$. We shall postpone the lengthy proof of the continuity of λ_U until the end of this chapter.

Now, observe that $t_{i-1}(\alpha) = t_i(\alpha)$ whenever $\alpha \notin W_{\mu_i}$. It follows that the value of λ_U at (e,α) depends only on the sets W_μ that contain α. In fact, if $\xi_1 < \xi_2 < \cdots < \xi_m$ are the indices of the sets W_μ that contain α, then letting $q_r(\alpha) = \sum_{i=1}^{r} C_{\xi_i}(\alpha)/\sum_{i=1}^{m} C_{\xi_i}(\alpha) \quad r=1,\ldots,m,$ we find that

$$\lambda_U(e,\alpha)(t) = \lambda_{\xi_1}(e,\alpha,o)(t) \qquad \text{if } o\leq t\leq q_1(\alpha)$$

$$= \lambda_{\xi_2}(\lambda_U(e,\alpha)(q_1(\alpha)),\alpha,q_1(\alpha))(t) \quad \text{if } q_1(\alpha)\leq t\leq q(\alpha)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

We conclude from this observation and the fact that M is well ordered that, if $(e,\alpha) \in \bar{B}_U \cap \bar{B}_V$, then $\lambda_U(e,\alpha) = \lambda_V(e,\alpha)$ for U,V in $\{U\}$. Therefore, since $\{\bar{B}_U(U \in \{U\}\}$ is an open covering of $\bar{B}$, and since $\lambda_U|\bar{B}_U \cap \bar{B}_V = \lambda_V|\bar{B}_U \cap \bar{B}_V$ for all $V,U \in \{U\}$ we can define a continuous λ: $\bar{B}\to E^I$ by setting $\lambda|\bar{B}_U = \lambda_U$. Clearly λ is the desired lifting function.

Definition 1. A map $p\colon E \to B$ is a <u>local fibration</u> if there is an open covering $\{U\}$ of B such that $p/p^{-1}(U)\colon p^{-1}(U) \to U$ is a fibration for each $U \in \{U\}$.

Corollary 1.1 Let $p\colon E \to B$ be a map and assume that B is paracompact. Then p is a fibration if and only if p is a local fibration.

Proof. It is clear that every fibration is a local fibration.

To prove the converse, let $\{U\}$ be the open covering of B of the definition. Since B is paracompact, $\{U\}$ has a nbd-finite refinement $\{V_\beta\}$. By Theorem 3 in Chapter 0 since B is normal, there is a nbd-finite refinement $\{W_\beta\}$ of $\{V_\beta\}$ composed of closed sets. Also there is is a continuous function $C_\beta\colon B \to I$ such that $C_\beta(W_\beta) = 1$ and $C_\beta(B-V_\beta) = 0$. Let $U_\beta = C_\beta^{-1}(I-\{0\})$. Then $U_\beta \subset V_\beta$ and hence, $p/p^{-1}(U_\beta)\colon p^{-1}\ U_\beta) \to U_\beta$ is a fibration since it is the restriction of a fibration. Also C_β is a characterizing function for U_β and therefore, U_β is a cozero set. Since $W_\beta \subset U_\beta \subset V\beta_1$ it follows that $\{U_\beta\}$ is nbd-finite. The hypothesis of Lemma 1 are now satisfied. Thus p is a fibration.

We end this chapter by providing a proof of the continuity of the function $\lambda_U\colon \bar{B}_U \to E^I$ as defined in Proposition 1, Chapter 3.

Let $(e,\alpha) \in \bar{B}_U$ and let (K,V) be a subbasic open set in E^I containing $\lambda_U(e,\alpha)$. Since λ_{μ_n} is continuous, there exist open sets $E_n \subset E$, $B_n \subset B^I$ and $I_n \subset I$ such that $\lambda_U(e,\alpha)(t_{n-1}(\alpha)) \in E_n$, $\alpha \in B_n$ and $t_{n-1}(\alpha) \in I_n$; and such that $\lambda_{\mu_n}(E_n \times B_n \times I_n \cap D_{\mu_n}) \subset (K\ \ [t_{n-1}(\alpha),1],V)$. By definition, then

$$\lambda_{\mu_{n-1}}[\lambda_U(e,\alpha)(t_{n-2}(\alpha)),\alpha,t_{n-2}(\alpha)](t_{n-1}(\alpha)) = \lambda_U(e,\alpha)(t_{n-1}(\alpha))$$

which is an element in E_n. Now, the composition $D_{\mu_{n-1}} \times I \xrightarrow{\lambda_{\mu_{n-1}} \times 1_I} E^I \times I \xrightarrow{E} E$ is continuous since $\lambda_{\mu_{n-1}}$ is continuous. Thus there are open sets $E_{n-1} \subset E$, $B_{n-1} \subset B^I$, $I_{n-1} \subset I$ and $O_{n-1} \subset I$ so that $\lambda_U(e,\alpha)(t_{n-2}(\alpha)) \in E_{n-1}$, $\alpha \in B_{n-1}$, $t_{n-2}(\alpha) \in I_{n-1}$ and $t_{n-1}(\alpha) \in O_{n-ij}$ and such that (again, since $\lambda_{\mu_{n-1}}$ is continuous)

$$\lambda_{\mu_{n-1}}(E_{n-1} \times B_{n-1} \times I_{n-1} \cap D_{\mu_{n-1}})(O_{n-1}) \subset E_n,$$

and

$$\lambda_{\mu_{n-1}}(E_{n-1} \times B_{n-1} \times I_{n-1} \cap D_{\mu_{n-1}}) \subset (K \quad [t_{n-2}(\alpha),t_{n-1}(\alpha)],\ V).$$

Since t_{n-1} is continuous, an open set $B'_{n-1} \subset B^I$ such that $\alpha \in B'_{n-1}$ and $t_{n-1}(B'_{n-1}) \subset O_{n-1} \cap I_n$. We then have:

$$\lambda_{\mu_{n-1}}(E_{n-1} \times B_{n-1} \cap B'_{n-1} \times I_{n-1} \cap D_{\mu_{n-1}}) \subset (K \cap [t_{n-2}(\alpha),t_{n-1}(\alpha)],V),$$

$$\lambda_{\mu_{n-1}}(E_{n-1} \times B_{n-1} \cap B'_{n-1} \times I_{n-1} \cap D_{\mu_{n-1}})(O_{n-1}) \subset E_n$$

and $t_{n-1}(B_{n-1} \cap B'_{n-1}) \subset I_n$. These are all the requirements needed. We may proceed inductively and obtain open sets E_1 containing e, B_1 and B'_1 containing α, and I, containing o such that $\lambda_{\mu_1}([E_1 \times B_1 \cap B'_1 \times I_n] \cap D_{\mu_1})(t_1(B_1 \cap B'_1)) \subset E_2$, $\lambda_{\mu_1}([E_1 \times B_1 \cap B'_1 \times I_1] \cap D_{\mu_1}) \subset (K \cap [o,t_1(\alpha)],V)$ and $t_1(B_1 \cap B'_1) \subset I_2$.

Note that the set O_{n-1} found above was only used as a tool in order to obtain B'_{n-1}. In the last step it is understood that B'_1 was obtained in the same manner.

Let $B = (\bigcap_{i=1}^{n-1} B_i \cap B'_i) \bigcap B_n$. From the construction above it follows that $e \in E_1$, $\alpha \in B$, $\lambda_U(E_1 \times B \bigcap \bar{B}_U) \subset (K,V)$ and whence λ_U is continuous.

BIBLIOGRAPHY

1. Dugundji, J., *Topology*. Boston, Mass.: Allyn and Bacon, 1965.

2. Hu, S.-T., *Homotopy Theory*. New York: Academic Press, Inc., 1959.

3. Spanier, E. H., *Algebraic Topology*. New York: McGraw-Hill Book Co., Inc., 1967.